International Journal of Latest Trends in Engineering and Technology
Vol.(10)Issue(1), pp.044-049
DOI: http://dx.doi.org/10.21172/1.101.08
e-ISSN:2278-621X

SOME RESULTS ON N-NORMS

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Abstract- In this paper, we introduce some n-norms on a linear space X, induced by n-norms on its dual space X'. Also, we prove their equality. In addition, we prove the equality of two n-norms on a linear space X- one induced by a norm on X' and other induced by an n-norm on X' where X' is the dual space of X.
Keywords- Norm, n-Norm, n-Normed Space, Dual Space.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

($2N_1$) $\|x,y\| = 0$ iff x and y are linearly dependent.
($2N_2$) $\|x,y\| = \|y,x\|$.
($2N_3$) $\|\alpha x,y\| = |\alpha|\|x,y\|\ \forall\ x,y \epsilon X$ and $\alpha \epsilon \mathbb{R}$.
($2N_4$) $\|x+y,z\| = \|x,y\| + \|y,z\|\ \forall\ x,y,z \epsilon X$.

Then, $\|.,.\|$ is called a 2-norm on X and $(X,\|.,.\|)$ is called a linear 2-normed space. 2-norms are non-negative and $\|x,y+\alpha x\| = \|x,y\|$ for every $x,y \epsilon X$ and $\alpha \epsilon \mathbb{R}$.

The concept of 2 -normed spaces was initially investigated and developed by S.Gähler in 1960s and has been extensively developed by Y.J. Cho, C. Diminnie, S.Gähler, A. White and many others [1,2,3,4,8,12].

Let X be a real vector space with dim $X \geq n$ where n is a positive integer. A real valued function $\|.,\ldots,.\|:X^n \to \mathbb{R}$ is called an n-norm on X if the following conditions hold:

(1) $\|x_1,\ldots,x_n\| = 0$ iff $x_1,\ldots x_n$ are linearly dependent.
(2) $\|x_1,\ldots,x_n\|$ remains invariant under permutations of $x_1,\ldots x_n$.
(3) $\|\alpha x_1,x_2,\ldots x_n\| = |\alpha|\,\|x_1,\ldots,x_n\|\ \forall\ x_1,\ldots x_n \epsilon X$ and $\alpha \epsilon \mathbb{R}$.
(4) $\|x_0+x_1,x_2,\ldots,x_n\| \leq \|x_0,\ldots,x_n\| + \|x_1,\ldots,x_n\|\ \forall\ x_0,x_1,\ldots x_n \epsilon X$.

The pair $(X,\|.,\ldots,.\|)$ is called an n-normed space.

Let X be a real vector space with dim$X \geq n$ and be equipped with an inner product $\langle.,.\rangle$. Then the standard n-norm on X is given by $\|x_1,\ldots,x_n\|_S = \sqrt{\det[\langle x_i,x_j\rangle]}$.

Note that the value of $\|x_1,\ldots,x_n\|_S$ represents the volume of n-dimensional parallelepiped spanned by $x_1,\ldots x_n$.

A standard example of n-normed space is $X = \mathbb{R}^n$ equipped with the Euclidean n-norm:

$$\|x_1,\ldots,x_n\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1},\ldots,x_{in}) \epsilon \mathbb{R}^n$ for each $i = 1,2,\ldots,n$.

If X is an n-normed space with dual X', the following formula formulated by Gähler [2]

$$\|x_1,\ldots,x_n\|^G = \sup_{f_j \epsilon X',\|f_j\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X.

Let X be a Hilbert space with dual X'. Then Gähler's n-norm on X becomes

$$\|x_1,\ldots,x_n\|^G = \sup_{y_j \epsilon X',\|y_j\| \leq 1} \det[\langle x_i,y_j\rangle].$$

Also the function

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$$\|x_1,\dots,x_n\|^D = \sup_{y_j\in X, \|y_1,\dots,y_n\|^S\le 1} \begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle \end{vmatrix}$$

defines an n-norm on a Hilbert space X. If X is a seperable Hilbert space and $\{e_1, e_2, \dots\}$ is a complete orthonormal set in X, we can define an n-norm on X as

$$\|x_1,\dots,x_n\|_2 = \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|det|\alpha_{ij}|\right|^2\right]^{\frac{1}{2}}$$

where $\alpha_{ij} = \langle x_i, e_j\rangle$ [5,7].

Further, the function

$$\|x_1,\dots,x_n\|^E = \sup_{y_j\in X, \|y_1,\dots,y_n\|^S = 1} \begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle \end{vmatrix}$$

defines an n-norm on a Hilbert space X and the function

$$\|x_1,\dots,x_n\|^r = \sup_{f_j\in X', \|f_j\|=1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on a normed space X with dual X' [9]. If X is a Hilbert space, $\|x_1,\dots,x_n\|^r$ becomes $\|x_1,\dots,x_n\|^r = \sup_{y_j\in X', \|y_j\|=1} \det[\langle x_i, y_j\rangle]$.

The function

$$\|x_1,\dots,x_n\|^F = \sup_{y_j\in X, \|y_1,\dots,y_n\|^S\neq 0} \frac{\begin{vmatrix} \langle x_1,y_1\rangle & \cdots & \langle x_1,y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,y_1\rangle & \cdots & \langle x_n,y_n\rangle \end{vmatrix}}{\|y_1,\dots,y_n\|^S} \text{ [10]}$$

defines an n-norm on a Hilbert space X and the function

$$\|x_1,\dots,x_n\|^H = \sup_{f_j\in X', \|f_j\|\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\|\cdots\|f_n\|}$$

defines an n-norm on a normed space X with dual X'. On a Hilbert space X with dual X', $\|x_1,\dots,x_n\|^H$ becomes

$$\|x_1,\dots,x_n\|^H = \sup_{y_j\in X', \|y_j\|\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|y_1\|\|y_2\|\cdots\|y_n\|}.$$

Then, $\|.,\dots,.\|^D$, $\|.,\dots,.\|^E$, $\|.,\dots,.\|^F$, $\|.,\dots,.\|^G$, $\|.,\dots,.\|^H$, $\|.,\dots,.\|^r$ and $\|.,\dots,.\|^S$ are identical on a Hilbert space [11].

Let $(X, \|.,\dots,.\|)$ be an n-normed space. The following real valued functions on $(X')^n$ where X' = dual of :

$$\|f_1,\dots,f_n\|' = \sup_{x_i\in X, \|x_1,\dots,x_n\|\le 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix},$$

$$\|f_1,\dots,f_n\|'_1 = \sup_{x_i\in X, \|x_1,\dots,x_n\| = 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix},$$

$$\|f_1,\dots,f_n\|'_2 = \sup_{x_i\in X, \|x_1,\dots,x_n\|\neq 0} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines n-norms on X' and are identical [9,11].

2. MAIN RESULTS

Let X be a linear space with $\dim X \ge n$ where n is a positive integer and $\|.,\dots,.\|'$ be an n-norm defined on its dual X'.

PROPOSITION 2.1. The function $\|x_1,\dots,x_n\|_a : X^n \to \mathbb{R}$ given by

$$\|x_1,\dots,x_n\|_a = \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

defines an n-norm defined on X.

Proof: (i) $x_1,\dots,x_n$ are linearly dependent.

$\Leftrightarrow$ Rows of $det[f_j(x_i)]$ are linearly dependent.

$\Leftrightarrow \|x_1,x_2,\dots,x_n\| = 0$.

(ii) $\|x_1,\dots,x_n\|_a$ remains invariant under the permutations of $x_1,x_2,\dots,x_n$.

(iii) For $\alpha\epsilon\mathbb{R}$,

$$\|\alpha x_1,\dots,x_n\|_a = \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(\alpha x_1) & \cdots & f_n(\alpha x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$= \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\alpha\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$= |\alpha| \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$= |\alpha|\,\|x_1,\dots,x_n\|_a$$

(iv) For $x_0,x_1,\dots,x_n\epsilon X$,

$$\|x_0+x_1,\dots,x_n\|_a = \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_0+x_1) & \cdots & f_n(x_0+x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$\leq \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$+ \sup_{f_i\in X', \|f_1,\dots f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

Therefore, $\|x_0+x_1,\dots,x_n\|_a \leq \|x_0,\dots,x_n\|_a + \|x_1,\dots,x_n\|_a$. This completes the proof.

PROPOSITION 2.2. The function $\|x_1,\dots,x_n\|_b : X^n \to \mathbb{R}$ given by

$$\|x_1,\dots,x_n\|_b = \sup_{f_i\in X, \|f_1,\dots,f_n\|=1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

defines an n-norm on X.

PROPOSITION 2.3. The function $\|x_1,\dots,x_n\|_c : X^n \to \mathbb{R}$ given by

$$\|x_1,\dots,x_n\|_b = \sup_{f_i\in X, \|f_1,\dots,f_n\|\leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

defines an n-norm on X.

THEOREM 2.4. If the dual X' of a linear space X is an n-normed space, then the linear space X is also an n-normed space.

Remark: the n-norm of the dual space X' always induces an an n-norm on X.

PROPOSITION 2.5. On a linear space X with its dual X' and $\dim X \geq n$ where n is a positive integer, the n-norms $\|.,\dots,.\|_a$ and $\|.,\dots,.\|_b$ are identical.

Proof: Let $\|.,\dots,.\|'$ be an n-norm defined on X'. Then

$$\|x_1,\dots,x_n\|_a = \sup_{f_i\in X',\|f_1,\dots,f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

and

$$\|x_1,\dots,x_n\|_b = \sup_{f_i\in X',\|f_1,\dots,f_n\|'=1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}.$$

Clearly, $\|x_1,\dots,x_n\|_b \leq \|x_1,\dots,x_n\|_a$.

Conversely, define

$$g_i = \frac{f_i}{\sqrt[n]{\|f_1,\dots f_n\|'}}, \|f_1,\dots,f_n\|' \neq 0$$

$$= \frac{f_i}{\gamma}, \gamma^n = \|f_1,\dots,f_n\|'.$$

Clearly, $g_i \in X'$ for each $i = 1,2,\dots,n$ and $\|g_1,\dots g_n\|' = 1$.

Now,

$$\frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'} = \frac{\begin{vmatrix} \gamma g_1(x_1) & \cdots & \gamma g_n(x_1) \\ \vdots & \ddots & \vdots \\ \gamma g_1(x_n) & \cdots & \gamma g_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$= \frac{\gamma^n \begin{vmatrix} \gamma g_1(x_1) & \cdots & \gamma g_n(x_1) \\ \vdots & \ddots & \vdots \\ \gamma g_1(x_n) & \cdots & \gamma g_n(x_n)\end{vmatrix}}{\|f_1,\dots f_n\|'}$$

$$= \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}$$

$$\leq \sup_{g_i\in X',\|g_1,\dots,g_n\|'=1} \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}$$

$$= \|x_1,\dots,x_n\|_b \quad \forall\ f_i \in X',\ \|f_1,\dots,f_n\|' \neq 0$$

$$\Rightarrow \sup_{f_i\in X',\|f_1,\dots,f_n\|'\neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots,f_n\|'} \leq \|x_1,\dots,x_n\|_b$$

$\Rightarrow \|x_1,\dots,x_n\|_a \leq \|x_1,\dots,x_n\|_b$.

This completes the proof.

PROPOSITION 2.6. On a linear space X, the n-norms $\|.,\dots,.\|_b$ and $\|.,\dots,.\|_c$ are identical.

Proof: Let $\|.,\dots,.\|'$ be an n-norm defined on X'. Then,

$$\|x_1,\dots,x_n\|_b = \sup_{f_i\in X',\|f_1,\dots,f_n\|'=1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}$$

and

$$\|x_1,\dots,x_n\|_c = \sup_{f_i\in X',\|f_1,\dots,f_n\|'\leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}.$$

Clearly, $\|x_1,\dots,x_n\|_b \leq \|x_1,\dots,x_n\|_c$.

Conversely, define

$$g_i = \frac{f_i}{\sqrt[n]{\|f_1,\dots,f_n\|'}}, 0 < \|f_1,\dots f_n\|' \leq 1$$

$= \frac{f_i}{\gamma}, \gamma^n = \|f_1, \dots, f_n\|'$.

Now,

$$\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} \gamma g_1(x_1) & \cdots & \gamma g_n(x_1) \\ \vdots & \ddots & \vdots \\ \gamma g_1(x_n) & \cdots & \gamma g_n(x_n) \end{vmatrix}$$

$$= \gamma^n \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}$$

$$\le \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix} \quad (\because 0 < \gamma^n \le 1)$$

$$\le \sup_{g_i \in X', \|g_1, \dots, g_n\|' = 1} \begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{vmatrix}$$

$$= \|x_1, \dots, x_n\|_b \ \forall\ f_i \in X',\ 0 < \|f_1, \dots, f_n\|' \le 1$$

$$\Rightarrow \sup_{f_i \in X', \|f_1, \dots, f_n\|' \le 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} \le \|x_1, \dots, x_n\|_b$$

$\Rightarrow \|x_1, \dots, x_n\|_c \le \|x_1, \dots, x_n\|_b$.

This completes the proof.

Cor. On a linear space X with its dual X' and $\dim X \ge n$ where n is a positive integer, the n-norms $\|.,\dots,.\|_a, \|.,\dots,.\|_b$ and $\|.,\dots,.\|_c$ are identical.

PROPOSITION 2.8 Let X be a real vector space with $\dim X \ge n$ where n is a positive integer and X' be its dual. Also, let $(X, \|.\|)$ and $(X, \|.,\dots,.\|)$ be respectively a normed space and an n-normed space. Then the following the n-norms defined on X:

$$\|x_1, \dots, x_n\|^H = \sup_{f_j \in X', \|f_j\| \ne 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1\|\|f_2\| \cdots \|f_n\|}$$

and

$$\|x_1, \dots, x_n\|_a = \sup_{f_j \in X', \|f_1, \dots, f_n\|' \ne 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}}{\|f_1, \dots, f_n\|'}$$

are identical.

Proof: Define

$$g_i = \frac{\|g_i\| f_i}{\sqrt[n]{\|f_1, \dots, f_n\|'}}, \|f_1, \dots, f_n\|' \ne 0$$

$$= \frac{\|g_i\| f_i}{\gamma}, \gamma^n = \|f_1, \dots f_n\|'$$

Then,

$$\|g_1, \dots, g_n\|' = \left\|\frac{\|g_1\| f_1}{\gamma}, \dots, \frac{\|g_n\| f_n}{\gamma}\right\|'$$

$$= \frac{\|g_1\|\|g_2\| \dots \|g_n\|}{\gamma^n} \|f_1, \dots, f_n\|'$$

$$= \|g_1\|\|g_2\| \dots \|g_n\|$$

Now,

$$\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} \frac{\gamma g_1}{\|g_1\|}(x_1) & \cdots & \frac{\gamma g_n}{\|g_n\|}(x_1) \\ \vdots & \ddots & \vdots \\ \frac{\gamma g_1}{\|g_1\|}(x_n) & \cdots & \frac{\gamma g_n}{\|g_n\|}(x_n) \end{vmatrix}$$

$$= \frac{\gamma^n}{\|g_1\|\|g_2\|\dots\|g_n\|}\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}$$

$$= \frac{\|f_1,\dots,f_n\|'}{\|g_1\|\|g_2\|\dots\|g_n\|}\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}$$

$$\Rightarrow \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots,f_n\|'} = \frac{1}{\|g_1\|\|g_2\|\dots\|g_n\|}\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix} \qquad (2.8.1)$$

$$\leq \operatorname*{Sup}_{g_j \epsilon X', \|g_j\| \neq 0} \frac{\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}}{\|g_1\|\|g_2\|\cdots\|g_n\|}$$

$$\Rightarrow \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots,f_n\|'} \leq \|x_1,\dots,x_n\|^H \quad \forall\ f_j \epsilon X' \text{ with } \|f_1,\dots f_n\|' \neq 0$$

$$\Rightarrow \operatorname*{Sup}_{f_j \epsilon X', \|f_1,\dots,f_n\|' \neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots,f_n\|'} \leq \|x_1,\dots,x_n\|^H$$

$$\Rightarrow \|x_1,\dots,x_n\|_a \leq \|x_1,\dots,x_n\|^H. \qquad (2.8.2)$$

Again taking on left side of $(2.8.1)$ over all $f_i s$ with $\|f_1,\dots f_n\|' \neq 0$, we have

$$\operatorname*{Sup}_{f_j \epsilon X', \|f_1,\dots,f_n\|' \neq 0} \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{vmatrix}}{\|f_1,\dots,f_n\|'} \geq \frac{1}{\|g_1\|\|g_2\|\dots\|g_n\|}\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix}$$

$$\Rightarrow \|x_1,\dots,x_n\|_a \geq \frac{1}{\|g_1\|\|g_2\|\dots\|g_n\|}\begin{vmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n)\end{vmatrix} \forall\ g_j \epsilon X' \text{ with } \|g_j\| \neq 0$$

$$\Rightarrow \|x_1,\dots,x_n\|_a \geq \|x_1,\dots,x_n\|^H \qquad (2.8.3)$$

From $(2.8.2)$ and $(2.8.3)$, the conclusion follows. This completes the proof.

3. CONCLUSION

The above n-norms and equalities will be helpful in the study of n-linear functional, n-linear operators and other related fields.

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